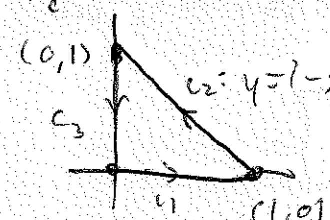
Ch. 8 Classical Integration Theorems of Vector Calculus

Sec 8.1 **Green's Theorem**: Let $\vec{c}$ be a positively oriented (= counterclockwise) piecewise smooth simple closed curve in the plane, and let D be the region bounded by $\vec{c}$. If $F(x,y) = (P(x,y), Q(x,y))$ is a C^1 vector field on D, then

$$\boxed{\underbrace{\int_{\vec{c}} F \cdot d\vec{s} = \int_{\vec{c}} P\,dx + Q\,dy}_{\text{path integral of } F \text{ along } \vec{c} = \partial D} = \underbrace{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (\nabla \times F) \cdot \vec{k}\, dA}_{\text{double integral of scalar curl of } F \text{ over } D}}$$

Example (evaluating path integrals with Green's Theorem)

$\int_{\vec{c}} x^4\,dx + xy\,dy = \int_{\vec{c}} (x^4, xy) \cdot d\vec{s}$, where $\vec{c}$ is as shown:



Note: Computing the path integral would require 3 parametrizations and 3 integrals.

Soln
$$\int_{\vec{c}} x^4\,dx + xy\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$
$$= \int_0^1 \int_0^{1-x} \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^4)\,dy\,dx$$
$$= \int_0^1 \int_0^{1-x} y\,dy\,dx$$
$$= \int_0^1 \left[\frac{1}{2}y^2\right]_0^{1-x} dx$$
$$= \frac{1}{2}\int_0^1 (1-x)^2\,dx$$
$$= \frac{-1}{6}\left[(1-x)^3\right]_0^1 = \boxed{\frac{1}{6}}.$$

- Using Green's Theorem to compute areas:

area of $D = \iint_D dA$, so if we choose $F = (P, Q)$ so that

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, we have $\boxed{\text{area of } D = \iint dA = \int_{\vec{c}} P\,dx + Q\,dy = \int_{\vec{c}} F \cdot d\vec{s}}$

Common formulas include:

$$\boxed{\text{area of } D = \int_{\vec{c}} x\,dy = -\int_{\vec{c}} y\,dx = \frac{1}{2}\int_{\vec{c}} -y\,dx + x\,dy}$$

<u>Example</u> Area of an ellipse $\vec{c}(t) = (a\cos t, b\sin t)$, $t \in [0, 2\pi]$

<u>Soln</u> The last formula is often simple with trig functions.

$$\text{area} = \frac{1}{2}\int_{\vec{c}} -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} (-b\sin t)(-a\sin t) + (a\cos t)(b\cos t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,dt = \boxed{\pi ab}.$$

- Green's Theorem is the "FTC for double integrals":

<u>Recall</u>: FTC: $\int_a^b f'(x)\,dx = f(b) - f(a) \leftarrow$ how to compute integral using info at the boundary of $[a, b]$

- Green's Theorem: how to compute double integral over D using info at the boundary of D, $\partial D = \vec{c}$.

Sec 8.3 Gauss' Divergence Theorem: Let E be a simple solid region in $\mathbb{R}^3$ and let S be the boundary surface of E. Let F be a C^1 vector field on E with the positive orientation (= outward).

Then

$$\underbrace{\iint_S F \cdot d\vec{S}}_{\substack{\text{surface integral/} \\ \text{flux of } F \text{ across } S = \partial E}} = \underbrace{\iiint_E \operatorname{div} F \, dV = \iiint_E \nabla \cdot F \, dV}_{\text{triple integral of divergence of } F \text{ over } E}$$

- Analog of the FTC in that it says how to compute an integral over a solid region using info on its boundary.

Example $\iint_S F \cdot d\vec{S}$, where $F = (z, y, x)$, S = unit sphere

Soln $\nabla \cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (z, y, x) = 0 + 1 + 0 = 1$, so

$$\iint_S F \cdot d\vec{S} = \iiint_E \nabla \cdot F \, dV, \quad \text{where } E = \text{unit ball}$$

$$= \iiint dV = \text{volume of } E = \boxed{\tfrac{4}{3}\pi(1^3)}$$

• Divergence Theorem in the plane: Let $F=(P,Q)$ be a C^1 vector field on $D \subseteq \mathbb{R}^2$. Let $\vec{c}=\partial D$ be the boundary of D, with the positive orientation (= counterclockwise). Let $\vec{N}$ be the outward unit normal to $\vec{c}$. Then

$$\underbrace{\int_{\vec{c}} F \cdot \vec{N}\, ds = \int_{\vec{c}} -Q\,dx + P\,dy}_{\text{outward flux of } F \text{ across } D} = \underbrace{\iint_D (\nabla \cdot F)\, dA}_{\text{double integral of divergence}}$$

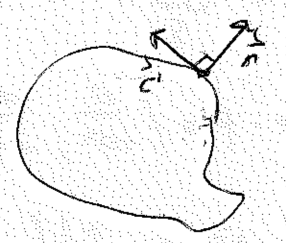
outward flux of F across D = double integral of divergence
= net area that flows out of D per unit time   of F over D.

Claim: Divergence Theorem in the plane is equivalent to Green's Theorem

Proof: $c'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$. An outward normal points $90°$ to the right, say $\vec{n} = \left(\frac{dy}{dt}, -\frac{dx}{dt}\right)$.

Then $\|\vec{n}\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{ds}{dt}$, so

$\left(\cancel{\frac{dy}{dt}}, \cancel{\frac{-dx}{dt}}\right) ds = \frac{\vec{n}}{\|\vec{n}\|} = \vec{N}\, ds$.



① LHS of Green's Theorem:

$$\int_{\vec{c}} P\,dx + Q\,dy = \int_{\vec{c}} (Q,-P)\cdot(dy,-dx)\, d\vec{s} = \int_{\vec{c}} (Q,-P)\cdot \vec{N}\, ds.$$

② Apply Divergence Theorem in the plane to get RHS of Green's Theorem:

$$\int_{\vec{c}} (Q,-P)\cdot\vec{N}\,ds = \iint_D (\nabla\cdot F)\, dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Combining ① and ②, we've shown:

$$\underbrace{\int_{\vec{c}} P\,dx + Q\,dy}_{\text{Green}} = \underbrace{\int_{\vec{c}} (Q,-P)\cdot\vec{N}\,ds}_{\text{Div}} = \underbrace{\iint_D (\nabla\cdot F)\,dA}_{\text{Div}} = \underbrace{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}_{\text{Green}}$$

<u>Example</u> Verify the Divergence Theorem in the Plane for

$F = (x^2, xy)$, $\vec{c}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$.

<u>Soln</u> $P = x^2 = \cos^2 t$ $\qquad Q = xy = \cos t \sin t$

$dx = -\sin t \, dt \qquad dy = \cos t \, dt$

so $\displaystyle\int_{\vec{c}} F \cdot \vec{N} \, ds = \int_{\vec{c}} -Q \, dx + P \, dy$

$$= \int_0^{2\pi} \left[(-\cos t \sin t)(-\sin t) + \cos^2 t(\cos t)\right] dt$$

$$= \int_0^{2\pi} \cos t \sin^2 t + \cos t \cos^2 t \, dt$$

$$= \int_0^{2\pi} \cos t \, dt = \boxed{0}$$

while $\displaystyle\iint_D \nabla \cdot F \, dA = \iint_D \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xy) \, dA$

$D = \{x^2 + y^2 \leq 1\}$

$$= \iint_D 3x \, dA$$

$$= 3\int_0^{2\pi}\int_0^1 r\cos\theta \cdot r \, dr \, d\theta$$

$$= 3\left(\int_0^{2\pi} \cos\theta \, d\theta\right)\left(\int_0^1 r^2 \, dr\right]$$

$$= 3 \cdot 0 \cdot \frac{1}{3} = \boxed{0}$$

Sec 8.3 Stokes' Theorem: Let S be an oriented, piecewise smooth surface bounded by a simple closed piecewise smooth curve $\vec{c}$ with positive orientation. Let F be a C^1 vector field on S. Then

$$\boxed{\underbrace{\int_{\vec{c}} F \cdot d\vec{s}}_{\substack{\text{path integral of } F \\ \text{along } \vec{c} = \text{circulation}}} = \underbrace{\iint_S (\nabla \times F) \cdot d\vec{S}}_{\text{surface integral of the curl of } F \text{ across } S}}$$

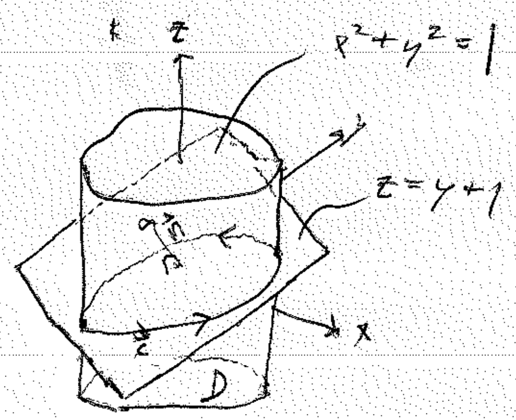
By definition, this is the same as

$$\boxed{\int_{\vec{c}} \vec{F} \cdot \vec{T} \, ds = \iint_S (\nabla \times F) \cdot \vec{N} \, dS}$$

where $\vec{T}$ = unit tangent vector, $\vec{N}$ = outward unit normal vector.

- Stokes' Theorem is an analog of the FTC in that is says how to compute a surface integral using info at the boundary of the surface.

Example $\int_{\vec{c}} F \cdot d\vec{s}$, where $F = (4z, -2x, 2x)$ and $\vec{c}$ is the intersection of the cylinder $x^2+y^2=1$ and the plane $z=y+1$, oriented counterclockwise.



Soln $\nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4z & -2x & 2x \end{vmatrix} = (0, 2, -2)$

so $\int_{\vec{c}} F \cdot d\vec{s} = \iint_S (\nabla \times F) \cdot d\vec{S}$

$= \iint_D (\nabla \times F) \cdot (\vec{r}_u \times \vec{r}_v)\, dA$

where $\vec{r}(u,v)$ is a counterclockwise parametrization of S and D is the disk $u^2 + v^2 \leq 1$.

Let $\vec{r}(u,v) = (u, v, \underbrace{v+1}_{z=y+1})$ for $u^2+v^2 \leq 1$

Then $\vec{r}_u \times \vec{r_v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = (0, -1, 1)$,

so $\int_{\vec{c}} F \cdot d\vec{s} = \iint_D (0, 2, -2) \cdot (0, -1, 1)\, dA$

$= -4 \iint_D dA$

$= -4(\text{area of } D) = -4\pi(1^2) = \boxed{-4\pi}$.

- Green's Theorem is a special case of Stokes' Theorem:

Green's Theorem: $\int_{\vec{c}} F \cdot d\vec{s} = \iint_D (\nabla \times F) \cdot \vec{k}\, dA$

Stoke's Theorem: $\int_{\vec{c}} F \cdot d\vec{s} = \iint_S (\nabla \times F) \cdot \vec{N}\, dS$